THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2050B Mathematical Analysis I (Fall 2016)
Tutorial Questions for 29 Sep

Remark: In this course, unless otherwise stated, you can almost never use the following functions, since they are not defined:

- Logarithmic functions and exponential functions.
- Trigonometric functions.
- Any non-elementary functions.

It is always safe to use polynomials and rational functions, however.

1. Show by definition that
$$\lim_{n\to\infty} \frac{1}{n^2} = 0$$
2. Show by definition that
$$\lim_{n\to\infty} \frac{5n^2+2n+3}{n^2+n+2} = 5$$
3. Show by definition that
$$\lim_{n\to\infty} \frac{n^2+1}{n^3-n^2-1} = 0$$
4. Let $A \subseteq \mathbb{R}$ be nonempty and bounded above. By the completeness axiom, $s := \sup A$ exists in $\mathbb{R}$.
   (a) Show that there exists a sequence $\{a_n\} \subseteq A$ such that (a_n) converges to s.
   (b) Show that, in addition, the above sequence can be taken to be monotonically increasing (not necessarily strictly increasing).
   (c) Show that, if we assume $s \notin A$ in addition, then the above sequence can be taken to be strictly increasing. (Also, think about a trivial example where we CANNOT find such strictly increasing sequence)
5. (Difficult) Let $-1 < r < 1$, and $k \in \mathbb{N}$. Show by definition that
$$\lim_{n\to\infty} n^k r^n = 0$$
   Remark: The above limit provides a useful (except in this course) heuristic that every polynomial grows slower than any (growing) exponential function eventually. Also, it is needed when we calculate sums of the form
$$\sum_{n=1}^{\infty} P(n) \cdot r^n,$$
   where P is a polynomial and $-1 < r < 1$.